Pergamon

J. Appl. Maths Mechs, Vol. 57, No. 4, pp. 591–600, 1993

0021–8928/93 $24.00+0.00

0021-8928(93)E0003-F

QUASISTATIC TREATMENT OF STABILITY FOR SOLUTIONS OF A CLASS OF MECHANICAL SYSTEMS WITH AN INFINITE NUMBER OF DEGREES OF FREEDOM†

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(*Received* 9 *December* 1992)

The stability of solutions of a certain class of singularly perturbed differential equations in Banach space, encountered in the theory of dynamics of deformed systems, are investigated. It is shown that in certain cases, when the equations of quasistatic motion of the system conform to criteria of asymptotic stability and instability in the first approximation, it suffices to investigate the stability of the solutions of these quasistatic equations. It is shown that the relative equilibrium of an inextensible viscoelastic ring, circumscribing a circular orbit in a plane orthogonal to the radius-vector of its centre of mass, is unstable.

1. STATEMENT OF THE PROBLEM

A "QUASISTATIC APPROACH"‡ to studying the dynamics of large elastic systems in a gravitational field has been developed. The method may be described as follows. The systems are simulated by continuous elastic bodies having internal damping within the framework of linear viscoelasticity theory. It is assumed that the body is fairly rigid and that the decay time of the free elastic vibrations is much less than the characteristic time of the body's motion as a whole. The field of displacements is sought as a series in the natural modes of free elastic vibrations of the body. On the basis of these assumptions, a small parameter is introduced and the system of equations for the dynamics of the system becomes a denumerable system of singularly perturbed equations

$$\dot{y}_s = f_s(y_j, q_n, \dot{q}_n), \quad s = 1, \ldots, k \tag{1.1}$$

$$\ddot{q}_n + 2\epsilon^{-1} b\omega_n^2 \dot{q}_n + \epsilon^{-2}\omega_n^2 q_n = Q_n(y_s, q_i, \dot{q}_i) \tag{1.2}$$

$$n = 1, 2, \ldots; \; 0 < \epsilon \ll 1$$

where y_s are the phase coordinates, which describe the motion of a trihedron attached to the body, q_n are generalized (normal) coordinates describing the deformations of the body, and ϵ is a small parameter characterizing the "high" rigidity of the body and the smallness of dissipative forces compared with elastic ones; the dot denotes differentiation with respect to time. The quantities $\epsilon^{-1}\omega_n$ are the natural frequencies of elastic vibrations ($b = 0$) of the body.

†*Prikl. Mat. Mekh.* Vol. 57, No. 4, pp. 12–20, 1993.

‡KLIMOV D. M. and MARKEYEV A. P., Non-linear problems of the dynamics of large-scale cosmic structures. Preprint No. 449, Inst. Problem Mekh. Akad. Nauk SSSR, Moscow, 1990

Asymptotic forms of the solutions of system (1.1), (1.2) can be constructed by the boundary-layer method [1] similar to the method developed in [2, 3] for systems with elastic and dissipative elements. The terms included in the asymptotic expressions for the generalized coordinates q_n correspond to quasistatic vibrations of the body and have the following form [4]

$$q_n = \epsilon^2 \omega_n^{-2} (Q_n(y_s, 0, 0) - 2\epsilon b Q_n^{\cdot}(y_s, 0, 0)) \tag{1.3}$$

where differentiation with respect to time is subject to the equations $y_s^{\cdot} = f_s(y_j, 0, 0)$. After substituting (1.3) into the finite-dimensional system (1.1) and dropping terms $O(\epsilon^4)$, the system of equations describing the motion of a trihedron attached to the body is closed and easier to handle; these are the equations of quasistatic motion.

Several workers [5–9]† have studied the stability of solutions of the equations of quasistatic motion. One as yet open question is the relationship between the stability/instability of their solutions and that of the solutions of the initial system (1.1), (1.2).

We will show below that if the linearized equations of quasistatic motion are asymptotically stable or unstable, the same is true of the full non-linear system.

2. THEOREMS ON STABILITY IN THE FIRST APPROXIMATION IN BANACH SPACE

We will generalize certain well-known stability theorems [10, 11].

Let E be a Banach space, in which we consider an autonomous differential equation

$$x^{\cdot} = Ax + F(x) \tag{2.1}$$

where A is a closed linear operator and F is a function satisfying the inequality

$$\|F(x)\| \leqslant N \|x\|^{1+p}, \quad p > 0, \quad N > 0 \tag{2.2}$$

in the domain $\|x\| \leqslant \rho$.

Let us consider the stability (in Lyapunov's sense) of the trivial solution of Eq. (2.1), assuming that the solutions possess the properties of existence, uniqueness and extendibility over an infinite time interval.

Let $\Sigma(A)$ denote the spectrum and $R(\xi, A) = (A - \xi)^{-1}$ the resolvent of A. Assume that: (1) A is a closed linear operator with domain dense in E; and (2) the semi-infinite interval $\xi > \beta$ is a subset of the resolvent set of A and

$$\|R(\mu + i\nu, A)\| \leqslant B(\mu - \beta)^{-1}, \ \mu - \beta, \ B = \text{const}$$

Under these two conditions, A generates a quasibounded semigroup [12]. Denote the set of all operators satisfying conditions 1 and 2 by $\Omega(B, \beta)$.

Theorem 1. Let $A \in \Omega(B, \beta)$ for $\beta < 0$ and suppose condition (2.2) is satisfied. Then the trivial solution of Eq. (2.1) is uniformly asymptotically stable.

Indeed, it follows from the assumptions of the theorem that the solutions of the first approximation equation

†See also: KLIMOV D. M., MARKEYEV A. P. and KHOLOSTOVA O. V., On the dynamics of an elastoviscous ring in a gravitational field. Preprint No. 406, Inst. Problem Mekh., Akad. Nauk SSSR, Moscow, 1989; KARPOV I. I., KLIMOV D. M. and MARKEYEV A. P., Analytical computer derivation of the equations of motion of an elastic body in a gravitational field. Preprint No. 411, Inst. Problem Mekh., Akad. Nauk SSSR, Moscow, 1989.

$$x^{\cdot} = Ax$$

satisfy the condition $\|x\| \leqslant Be^{\beta t}\|x(0)\|$; the proof may now be carried out by standard arguments [10, p. 51].

Now consider the case in which $\beta > 0$ and $\Sigma(A)$ contains points in the right half-plane.

Theorem 2. If: (1) $\Sigma(A)$ contains points in the right half-plane; and (2) $A \in \Omega(B, \beta)$ $(\beta > 0)$ and condition (2.2) is satisfied, then the trivial solution of Eq. (2.1) is unstable.

This generalizes a well-known result for bounded operators A [11, p. 410, Theorem 2.3]. Operators that satisfy the conditions of Theorem 2 possess the properties used to prove Theorem 2.3 of [11].

We will also need the following theorem.

Theorem 3 [12]. Let $A \in \Omega(B, \beta)$ and let C be a bounded linear operator. Then $A + C \in \Omega(B, \beta + B\|C\|)$.

3. STABILITY ANALYSIS

We will transform the system of equations (1.1), (1.2) by changing the variables [3] $q_n = \epsilon^2 q_n{}^*$, $q_n^{\cdot} = \epsilon p_n$ (the asterisk will be omitted henceforth) and putting

$$y = (y_1, \ldots, y_k),\ p = (p_1, p_2, \ldots),\ q = (q_1, q_2, \ldots)$$
$$C = \operatorname{diag}\{\omega_1^2, \omega_2^2, \ldots\},\ B = 2bC_{\ast}$$

System (1.1), (1.2) becomes

$$\begin{aligned} &y^{\cdot} = f(y, \epsilon^2 q, \epsilon p) \\ &p^{\cdot} = -\epsilon^{-1}Bp - \epsilon^{-1}Cq + \epsilon^{-1}Q(y, \epsilon p, \epsilon^2 q),\ q^{\cdot} = \epsilon^{-1}p \end{aligned} \tag{3.1}$$

Let us assume that system (3.1) has a stationary solution

$$y = y^0,\ p = 0,\ q = q^0 \tag{3.2}$$

in whose neighbourhood the right-hand sides of system (3.1) are twice continuously differentiable.

To investigate the stability of the solution (3.2), we consider the variational equations, retaining the same notation (y, p and q, respectively) for the variations of y, p and q

$$\begin{aligned} &y^{\cdot} = Ty + \epsilon Kp + \epsilon^2 Lq \\ &p^{\cdot} = (-\epsilon^{-1}B + B_1)p + (-\epsilon^{-1}C + \epsilon C_1)q + \epsilon^{-1}My,\ q^{\cdot} = \epsilon^{-1}p \end{aligned} \tag{3.3}$$

where T is a finite-dimensional operator (a k by k matrix) describing the motion of an absolutely rigid body, with a configuration corresponding to $q = q^0$, in the neighbourhood of (3.2). If the initial system (1.1) is Hamiltonian for $q = 0$, $q^{\cdot} = 0$, then either all the eigenvalues of T have zero real parts, or it has both eigenvalues with positive real parts and eigenvalues with negative real parts. The bounded operators K, L and M represent the relationships between the translational–rotational motion of the body and the deformation process; B_1 and C_1 are bounded operators resulting from the linearization of system (3.1) in the neighbourhood of (3.2).

System (3.3) may be treated as the first-approximation equation of Eq. (2.1) in the Banach space of sequence E

$$E = E_1 \oplus E_2 \oplus E_3,\ y \in E_1,\ p \in E_2,\ q \in E_3$$

We write (3.3) in matrix form

$$x^{\cdot} = Ax, \quad x = (y, p, q)^T$$

$$A = \begin{Vmatrix} T & \epsilon K & \epsilon^2 L \\ \epsilon^{-1} M & -\epsilon^{-1} B + B_1 & -\epsilon^{-1} C + \epsilon C_1 \\ 0 & \epsilon^{-1} I & 0 \end{Vmatrix}, \quad I = \operatorname{diag}\{1, 1, \ldots\} \tag{3.4}$$

The behaviour of the solutions in the neighbourhood of (3.2) depends on the properties of the operator A. To reduce (3.4) to a form more convenient for applying the theorems of Sec. 2, we change the variables $x \to x_1 = (y, \eta, \xi)^T$

$$p = \eta + \sum_{l=0}^{4} \epsilon^l \Lambda_l y, \quad q = \xi + \sum_{l=0}^{4} \epsilon^l \Gamma_l y \tag{3.5}$$

where the operators $\Lambda_i : E_1 \to E_2$, $\Gamma_i : E_1 \to E_3$ $(i = 0, \ldots, 4)$ are chosen so that the terms in the equations for $\eta^{\cdot}$ and $\xi^{\cdot}$ that depend on y are of the order ϵ^4.

Substituting (3.5) into the third equation of (3.3) and equating the coefficients of $\epsilon^i y$ $(i = 1, \ldots, 3)$ to zero, we obtain expressions for the operators Λ_i

$$\begin{aligned} &\Lambda_0 = 0, \quad \Lambda_1 = \Gamma_0 T, \quad \Lambda_2 = \Gamma_1 T \\ &\Lambda_3 = \Gamma_2 T + \Gamma_0 K \Lambda_1 + \Gamma_0 L \Gamma_0 \\ &\Lambda_4 = \Gamma_3 T + \Gamma_0 K \Lambda_2 + \Gamma_1 K \Lambda_1 + \Gamma_0 L \Gamma_1 + \Gamma_1 L \Gamma_0 \end{aligned} \tag{3.6}$$

Similarly, substituting (3.5) into the second equation of (3.3), we find that

$$\begin{aligned} &\Gamma_0 = C^{-1} M, \quad \Gamma_1 = -C^{-1} B C^{-1} M T \\ &\Gamma_2 = C^{-1} (B_1 \Lambda_1 + C_1 \Gamma_0 - B \Lambda_2 - \Lambda_1 T) \\ &\Gamma_3 = C^{-1} (B_1 \Lambda_2 + C_1 \Gamma_1 - B \Lambda_3 - \Lambda_2 T) \\ &\Gamma_4 = C^{-1} (B_1 \Lambda_3 + C_1 \Gamma_2 - B \Lambda_4 - \Lambda_3 T - \Lambda_1 K \Lambda_1) \end{aligned} \tag{3.7}$$

The operators Λ_i, Γ_i $(i = 0, \ldots, 4)$ are uniquely defined by (3.6) and (3.7) and are bounded, since C^{-1} and $C^{-1}B$ are bounded. It can be shown that the transformation $x \to x_1$ is an isomorphism, so that stability properties are preserved.

The new variables satisfy the equations

$$\begin{aligned} &y^{\cdot} = (T + \epsilon^2 (K\Lambda_1 + L\Gamma_0) + \epsilon^3 (K\Lambda_2 + L\Gamma_1)) y + \epsilon K \eta + \epsilon^2 L \xi + \epsilon^4 \Phi x_1 \\ &\eta^{\cdot} = (-\epsilon^{-1} B + B_1 + \sum_{l=0}^{2} \epsilon^{l+1} \Lambda_l K) \eta + (-\epsilon^{-1} C + \epsilon C_1 + \sum_{l=0}^{1} \epsilon^{l+2} \Lambda_l L) \xi + \epsilon^4 \theta x_1 \\ &\xi^{\cdot} = \epsilon^{-1} \eta - \sum_{l=0}^{2} \epsilon^{l+1} \Gamma_l K \eta - \sum_{l=0}^{1} \epsilon^{l+2} \Gamma_l L \xi + \epsilon^4 \Psi x_1 \end{aligned} \tag{3.8}$$

where Φ, θ and Ψ are bounded operators expressed in terms of Γ_i, Λ_i, K, L, ϵ.

The matrix form of the operator A_1 in system (3.8) may be written in the form

$$x_1^{\cdot} = A_1 x_1, \quad A_1 = A_1^0 + \epsilon^4 A_1^1$$

$$A_1^0 = \begin{Vmatrix} \tilde{T} & \epsilon K & \epsilon^2 L \\ 0 & \epsilon^{-1} R_1 + R_2 & \\ 0 & & \end{Vmatrix}, \quad R_1 = \begin{Vmatrix} -B & -C \\ I & 0 \end{Vmatrix}$$

$$A_1^1 = \Phi + \theta + \Psi, \quad \tilde{T} = T + \epsilon^2 (K\Lambda_1 + L\Gamma_0) + \epsilon^3 (K\Lambda_2 + L\Gamma_1)$$

where R_2 denotes terms of order at least ϵ^0 and at most ϵ^3 on the right-hand sides of the second and third equations of system (3.8).

We make yet another change of variables $x_1 \to x_2$, aimed at giving the operator of the system the following form in the new variables $x_2^{\bullet} = A_2 x_2$

$$A_2 = A_2^0 + \epsilon^4 A_2^1$$
$$A_2^0 = \begin{Vmatrix} \tilde{T} & 0 \\ 0 & \epsilon^{-1}R_1 + R_2 \end{Vmatrix}, \quad \| A_2^1 \| = \| A_1^1 \| \tag{3.9}$$

Changes of variable of this sort have been considered before for denumerable systems of differential equations [10]. All of them preserve stability.

Thus, the question of whether the solution (3.2) of system (3.1) is stable or not has been reduced to investigating the properties of the operator A_2 of (3.9), which is the sum of a closed operator A_2^0 and a bounded operator $\epsilon^4 A_2^1$. We will treat $\epsilon^4 A_2^1$ as a perturbation.

We will express A_2^0 as the direct sum of two operators defined in mutually complementary subspaces, E_1 and $E_2 \oplus E_3$: a finite-dimensional operator $\tilde{T}$ and a closed operator $\epsilon^{-1}R_1 + R_2$. For the former we have $\tilde{T} \in \Omega(1, \beta)$ in E_1 (β is the greatest real part of the eigenvalues of $\tilde{T}$). The operator $\epsilon^{-1}R_1$ characterizes the free damped oscillations of the damped system. It can be shown that $\epsilon^{-1}R_1 \in \Omega(1, -1/(2\epsilon))$. By Theorem 3, $\epsilon^{-1}R_1 + R_2 \in \Omega(1, -1/(2\epsilon) + \| R_2 \|)$ in $E_2 \oplus E_3$ ($\| R_2 \| = O(1)$). Consequently, $A_2^0 \in \Omega(1, \beta_1)$, where $\beta_1 = \max\{\beta, -1/(2\epsilon) + \| R_2 \|\} = \beta$.

Let us assume first that all the eigenvalues of T have a real part of zero.

If all the eigenvalues of $\tilde{T}$ have negative real parts, then $\beta < 0$ and A_2^0 satisfies the conditions of Theorem 1.

The greatest real part of the eigenvalues must have the form $-\epsilon^3 \lambda_{\max}^-$ ($\lambda_{\max}^- > 0$), since the internal elastic forces in the expression for $\tilde{T}$ correspond to $O(\epsilon^2)$ terms and the dissipative terms to $O(\epsilon^3)$ terms. By Theorem 3, if

$$\epsilon < \lambda_{\max}^- \| A_2^1 \|^{-1} \tag{3.10}$$

the operator A_2 also satisfies the conditions of Theorem 1, so the solution (3.2) is uniformly asymptotically stable.

Let us assume now that $\tilde{T}$ has eigenvalues with positive real parts.

Let $\epsilon^3\lambda_1^+, \epsilon^3\lambda_2^+, \ldots, \epsilon^3\lambda_l^+$ ($\lambda_i^+ > 0$; $\lambda_i^+ \leqslant \lambda_j^+$ for $i < j$) denote the positive real parts of the roots of the characteristic equation of $\tilde{T}$. Let $\rho = \max_i\{(\lambda_{i+1}^+ - \lambda_i^+)\} > 0$. By the spectral resolution theorems [12] and Theorem 3, it can be shown that if

$$\epsilon < \tfrac{1}{2} \| A_2^1 \|^{-1} \max\{\lambda_1^+, \rho\} \tag{3.11}$$

the operator A_2 satisfies the conditions of Theorem 2 and the unperturbed motion is unstable.

In the case when T has eigenvalues with positive real parts, $\tilde{T}$ has $O(1)$ eigenvalues and the proof of instability is similar.

Thus, for sufficiently small ϵ (see (3.10) and (3.11)), the stability or instability of the solution (3.2) of system (3.1) may be determined by considering the stability of the finite-dimensional system

$$y^{\bullet} = \tilde{T} y \tag{3.12}$$

provided the matrix $\tilde{T}$ satisfies an asymptotic stability or instability criterion in the first approximation.

As it turns out, Eqs (3.12) are simply the equations of quasistatic motion in the neighbourhood of (3.2).

We note that (3.12) may be obtained by substituting the expressions

$$p = \epsilon\Lambda_1 y + \epsilon^2 \Lambda_2 y$$
$$q = \Gamma_0 y + \epsilon\Gamma_1 y \tag{3.13}$$

into the first equation of system (3.3). In view of our previous change of variables $q = \epsilon^2 q^*$, $q^{\cdot} = \epsilon p$ and formulae (3.6) and (3.7), we can rewrite (3.13) as

$$q = \epsilon^2 C^{-1}(My - \epsilon BC^{-1}(My)^{\cdot}), \quad p = q^{\cdot} \tag{3.14}$$

where the differentiation with respect to time takes place along the trajectories of the equation $y^{\cdot} = Ty$. Comparing (3.13) with (2.14) in [2], we conclude that (3.14) represents linearized asymptotic expansions of the generalized coordinates corresponding to quasistatic oscillations.

We collect the above results in a theorem.

Theorem 4. Assume that the linearized equations of quasistatic motion in the neighbourhood of the unperturbed motion conform to a criterion of asymptotic stability (instability). Then for sufficiently small ϵ (see (3.10) and (3.11)) the unperturbed motion is asymptotically stable (unstable).

Remark. Instead of the exact particular solution (3.2) one frequently uses an approximate solution of Eqs (1.1) and (1.2)

$$y = \tilde{y}^0, \quad q_n = \tilde{q}_n^0 = \epsilon^2 \omega_n^{-2} Q_n(\tilde{y}^0, 0, 0), \quad \dot{q}_n = 0 \tag{3.15}$$

As a rule, the connection between the solutions (3.15) and (3.2) is defined by the relations

$$y^0 = \tilde{y}^0, \quad q_n^0 = \tilde{q}_n^0 + O(\epsilon^4) \tag{3.16}$$

The case (3.15), (3.16) is encountered, for example, in connection with the relative motion of a viscoelastic body in a central Newtonian force field. If f_s and Q_n are twice continuously differentiable and (3.15) is true, then

$$T = T_0 + O(\epsilon^4), \quad L = L_0 + O(\epsilon^2), \quad K = K_0 + O(\epsilon^2)$$
$$M = M_0 + O(\epsilon^2)$$
$$T_0 = \frac{\partial f}{\partial y}(y^0, \tilde{q}^0, 0), \quad L_0 = \frac{\partial f}{\partial q}(y^0, 0, 0)$$
$$K_0 = \frac{\partial f}{\partial p}(y^0, 0, 0), \quad M_0 = \frac{\partial Q}{\partial y}(y^0, 0, 0)$$

Replacing the operators T, K, L and M in (3.12) by T_0, K_0, L_0 and M_0, respectively, does not affect the accuracy with which Eqs (3.11) and (3.13) were obtained. It can be shown that the equations obtained by substituting (1.3) into (1.1) and linearizing in the neighbourhood of the solution (3.14) are just Eq. (3.11) with the operator expressed in terms of T_0, K_0, L_0 and M_0.

The proof of Theorem 4 in this case involves some slight modifications (the quantity $\|A_2^1\|$). is changed).

We also note that such conditions as $\lambda_{\max}^- > 0$ and $\lambda_1^+ > 0$ define subdomains of asymptotic stability or instability in the parameter domain. At parameter values for which $\lambda_{\max}^- = O(\epsilon)$, $\lambda_1^+ = O(\epsilon)$, conditions (3.10) and (3.11) may fail to hold and Theorem 4 will not apply.

The case in which the equations of quasistatic motion represent some critical case of elasticity theory will not be discussed here.

To sum up: with the exception of a few cases, as indicated, we have rigorously proved the results of [5–9] (see also the papers cited in the second footnote).

4. THE STABILITY OF THE RELATIVE EQUILIBRIUM OF A VISCOELASTIC INEXTENSIBLE RING IN A CIRCULAR ORBIT

Consider a viscoelastic inextensible ring moving in a central Newtonian force field. We assume that the motion of the ring's centre of mass is independent of the motion about the centre of mass and that the orbit of the centre of mass is circular ($\omega_0 = 2\pi T^{-1}$, where T is the period of revolution of the centre of mass around the orbit). Some aspects of the motion of such a system have been examined before (cf. the first paper cited in the second footnote). The necessary notation and equations of motion are as follows: $Ox_1x_2x_3$ is a "mean" coordinate frame attached to the ring (O is the centre of mass of the ring and x_3 is its axis of symmetry); $OX_1X_2X_3$ is an orbital frame. The X_3 axis points along the radius-vector of the centre of mass relative to the attractive centre and the X_2 and X_1 axes lie respectively along the binormal to the orbit and its transversal, in the direction of the centre of mass. Denote the unit vectors along the X_1, X_2 and X_3 axes by α, β and γ, respectively (α_i, β_i and γ_i denote their projections on the x_i axes).

The displacement $\mathbf{u}(\mathbf{r}, t)$ of a point of the ring (with radius-vector $\mathbf{r}$ in the undeformed state) due to two-dimensional bending vibrations may be written as

$$\mathbf{u}(\mathbf{r}, t) = \sum_{n=2}^{\infty} (q_n^{(1)}\mathbf{U}_n^{(1)}(r) + q_n^{(2)}\mathbf{U}_n^{(2)}(r))$$

where $\mathbf{U}_n^{(1)}$, $\mathbf{U}_n^{(2)}$ ($n = 2, 3, \ldots$) is an orthonormal system of natural modes of vibration.

The equations of motion of the trihedron $Ox_1x_2x_3$ may be written as

$$\begin{aligned} &\mathbf{K}^{\cdot} + \tilde{\boldsymbol{\omega}} \times \mathbf{K} = 3\boldsymbol{\gamma} \times J\boldsymbol{\gamma} \\ &\mathbf{K} = J\tilde{\boldsymbol{\omega}} + \mathbf{K}_*, \quad \mathbf{K}_* = \sum_{n=2}^{\infty} (q_n^{(1)\cdot} q_n^{(2)} - q_n^{(1)} q_n^{(2)\cdot})\,\mathbf{e}_3 \\ &\tilde{\boldsymbol{\omega}} = (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)^T, \quad \mathbf{e}_3 = (0, 0, 1)^T \end{aligned} \tag{4.1}$$

where $\tilde{\omega}_i$ is the projection of the absolute angular velocity of the trihedron $Ox_1x_2x_3$ on the x_i axis, multiplied by ω_0^{-1}. The dot in (4.1) and elsewhere below stands for differentiation with respect to the dimensionless time variable $\tau = \omega_0 t$, and J is the inertia tensor of the ring for a point in the system $Ox_1x_2x_3$

$$J = J_0 + J_1 + J_2, \quad J_0 = \operatorname{diag}\{A, A, C\} \tag{4.2}$$

$$J_1 = 2q_2^{(1)} \begin{Vmatrix} H_2 & 0 & 0 \\ 0 & -H_2 & 0 \\ 0 & 0 & 0 \end{Vmatrix} + 2q_2^{(2)} \begin{Vmatrix} 0 & -H_2 & 0 \\ -H_2 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

$$J_2 = \sum_{n=2}^{\infty} \left\{ (q_n^{(1)2} + q_n^{(2)2}) \begin{Vmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{Vmatrix} + 2(q_n^{(1)} q_{n+2}^{(2)} + q_n^{(2)} q_{n+2}^{(1)}) \times \right.$$

$$\left. \times \begin{Vmatrix} 0 & L_n & 0 \\ L_n & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} + 2(q_n^{(1)} q_{n+2}^{(1)} - q_n^{(2)} q_{n+2}^{(2)}) \begin{Vmatrix} -L_n & 0 & 0 \\ 0 & L_n & 0 \\ 0 & 0 & 0 \end{Vmatrix} \right\}$$

$$H_2 = \frac{3a}{2}\sqrt{\frac{\pi}{10}}, \quad L_n = \frac{(n-1)(n+3)}{4\sqrt{(n^2+1)((n+2)^2+1)}}$$

where a is the radius of the undeformed ring, m is its mass, A and C are its equatorial and axial moments of inertia, respectively. For a thin ring $C = 2A$. For generality, the ratio $\alpha = C/A$ will be assumed to be arbitrary from now on ($0 \leqslant \alpha \leqslant 2$).

The equations describing the variation with time of the generalized coordinates $q_n^{(1)}$, $q_n^{(2)}$ are

$$\begin{aligned}
&q_n^{(1)\cdot\cdot} + 2\epsilon^{-1} b\omega_0 \omega_n^2 q_n^{(1)\cdot} + \epsilon^{-2}\omega_n^2 q_n^{(1)} = Q_n^{(1)}(\tilde{\omega}, \gamma) + 2q_n^{(2)\cdot}\tilde{\omega}_3 + q_n^{(2)}\tilde{\omega}_3^{\cdot} + \tfrac{1}{2} q_n^{(1)}(\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \\
&+ 2\tilde{\omega}_3^2 + 3\gamma_1^2 + 3\gamma_2^2 - 2) + (q_{n+2}^{(1)} L_n + q_{n-2}^{(1)} L_{n-2})(\tilde{\omega}_1^2 - \tilde{\omega}_2^2 + 3\gamma_2^2 - 3\gamma_1^2) + \\
&+ 2(q_{n+2}^{(2)} L_n + q_{n-2}^{(2)} L_{n-2})(\tilde{\omega}_1\tilde{\omega}_2 - 3\gamma_1\gamma_2) \\
&q_n^{(2)\cdot\cdot} + 2\epsilon^{-1} b\omega_0 \omega_n^2 q_n^{(2)\cdot} + \epsilon^{-2}\omega_n^2 q_n^{(2)} = Q_n(\tilde{\omega}, \gamma) - 2q_n^{(1)\cdot}\tilde{\omega}_3 - q_n^{(1)}\tilde{\omega}_3^{\cdot} + \\
&+ \tfrac{1}{2} q_n^{(2)}(\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + 2\tilde{\omega}_3^2 + 3\gamma_1^2 + 3\gamma_2^2 - 2) + (q_{n+2}^{(2)} L_n + q_{n-2}^{(2)} L_{n-2})(\tilde{\omega}_2^2 - \tilde{\omega}_1^2 + \\
&+ \gamma_1^2 - \gamma_2^2) + 2(q_{n+2}^{(1)} L_n + q_{n-2}^{(1)} L_{n-2})(\tilde{\omega}_1\tilde{\omega}_2 - 3\gamma_1\gamma_2) \\
&Q_2^{(1)} = H_2(\tilde{\omega}_1^2 - \tilde{\omega}_2^2 + 3\gamma_2^2 - 3\gamma_1^2), \quad Q_n^{(1)} = 0, \quad n = 3, 4, \ldots \\
&Q_2^{(2)} = H_2(-2\tilde{\omega}_1\tilde{\omega}_2 + 6\gamma_1\gamma_2), \quad Q_n^{(2)} = 0, \quad n = 3, 4, \ldots
\end{aligned} \tag{4.3}$$

In Eqs (4.3) for $n = 2$, 3, we must formally equate $q_{n-2}^{(1)}$, $q_{n-2}^{(2)}$ to zero. The number b characterizes the dissipative properties of the material; $\epsilon^{-1}\omega_n = \omega_0^{-1}\Omega_n$, where Ω_n are the natural frequencies of two-dimensional bending vibrations of the ring, and ϵ is a small parameter, introduced in the usual way [2, 4].

To close system (4.1), (4.2), we add the kinematic Poisson equations

$$\boldsymbol{\alpha}^{\cdot} = \boldsymbol{\alpha} \times \tilde{\boldsymbol{\omega}} - \boldsymbol{\gamma}, \quad \boldsymbol{\beta}^{\cdot} = \boldsymbol{\beta} \times \tilde{\boldsymbol{\omega}}, \quad \boldsymbol{\gamma}^{\cdot} = \boldsymbol{\gamma} \times \tilde{\boldsymbol{\omega}} + \boldsymbol{\alpha} \tag{4.4}$$

Equations (4.1), (4.2) and (4.4) have an exact particular solution

$$\begin{aligned}
&\tilde{\boldsymbol{\omega}} = (0, 1, 0)^T, \quad \boldsymbol{\gamma} = (0, 0, 1)^T, \quad \boldsymbol{\alpha} = (1, 0, 0)^T \\
&q_n^{(1)\cdot} = 0, \quad q_n^{(2)\cdot} = 0, \quad q_n^{(2)} = 0, \quad n = 2, 3, \ldots
\end{aligned} \tag{4.5}$$

This solution satisfies Eqs (4.1) and (4.4) for any values of $q_n^{(1)}$. It corresponds to relative equilibrium of the ring in the orbital coordinate frame with its plane orthogonal to the radius-vector of its centre of mass relative to the attracting centre.

The stationary values of the generalized coordinates $q_n^{(1)}$ are determined by the relations

$$\begin{aligned}
&(\epsilon^{-2}\omega_2^2 + \tfrac{1}{2}) q_2^{(1)} + L_2 q_4^{(1)} = -H_2 \\
&L_{2k-2} q_{2k-2}^{(1)} + (\epsilon^{-2}\omega_{2k}^2 + \tfrac{1}{2}) q_{2k}^{(1)} + L_{2k} q_{2k+2}^{(1)} = 0 \\
&q_{2k-3}^{(1)} = 0, \quad k = 3, 4, \ldots
\end{aligned} \tag{4.6}$$

We will write the solution of system (4.6) as a series

$$\begin{aligned}
&\tilde{q} = \epsilon^2\tilde{q}_1 + \epsilon^4\tilde{q}_2 + \epsilon^6\tilde{q}_3 + \ldots, \quad \tilde{q} = (q_2^{(1)}, q_4^{(1)}, \ldots)^T \\
&\tilde{q}_1 = (-H_2\omega_2^{-2}, 0, 0, \ldots)^T, \quad \tilde{q}_i = \tilde{C}^{-1}\tilde{C}_1\tilde{q}_{i-1} \\
&\tilde{C} = \operatorname{diag}\{\omega_2^2, \omega_4^2, \omega_6^2, \ldots\}, \quad \tilde{C}_1 = \begin{Vmatrix} \tfrac{1}{2} & L_2 & 0 & \ldots \\ L_2 & \tfrac{1}{2} & L_4 & \ldots \\ 0 & L_4 & \tfrac{1}{2} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{Vmatrix}
\end{aligned} \tag{4.7}$$

Since $\tilde{C}^{-1}\tilde{C}_1$ is a compact operator, the series (4.7) converges absolutely if $\epsilon^2 \|\tilde{C}^{-1}\tilde{C}_1\| < 1$. Thus, condition (3.15) is satisfied and, by the remark in Sec. 3, we can use the approximate solution of system (4.6)

$$q_2^{(1)} = -\epsilon^2\omega_2^{-2}H_2, \text{ and the remaining } q_n^{(i)} = 0, \; i = 1, 2; \; n = 2, 3, 4, \ldots \tag{4.8}$$

We will now examine the stability of the particular solution (4.5), (4.7). The stability of the other two relative equilibria (when the plane of the ring lies in the orbital plane and when it is orthogonal to the velocity vector of the radius-vector of the centre of mass) has already been investigated (see the first paper cited in the second footnote).

Put

$$\tilde{\omega}_1 = y_1, \quad \tilde{\omega}_2 = 1 + y_2, \quad \tilde{\omega}_3 = y_3$$
$$\gamma_1 = y_4, \quad \gamma_2 = y_5, \quad \gamma_3 = 1 \tag{4.9}$$
$$\alpha_1 = 1, \quad \alpha_2 = y_6, \quad \alpha_3 = -y_4$$

These relations take into account that the vectors **α**, **β** and **γ** are orthonormal and that only three of the quantities α_i, β_i, γ_i $(i = 1, 2, 3)$ are independent.

Omitting the intermediate steps, we will merely give the linearized equations of quasistatic motion in the neighbourhood of the solution (4.5), (4.9) (for details of the procedure to be followed in deriving such equations see, e.g., [5])

$$(1 - \epsilon^2\kappa)\dot{y}_1 = -\epsilon^3\Delta(1-\alpha)y_1 + ((1-\alpha) + \epsilon^2\kappa(2\alpha - 1))y_3 +$$
$$+ (-3(1-\alpha) - 3\epsilon^2\kappa(2\alpha - 1))y_5 - \epsilon^3\Delta(1-\alpha)y_6 \tag{4.10}$$
$$(1 + \epsilon^2\kappa)\dot{y}_2 = -\epsilon^3\Delta(1-\alpha)y_2 + (3(1-\alpha) - 3\epsilon^2\kappa(2\alpha - 1))y_4$$
$$\alpha\dot{y}_3 = -1/3\,\epsilon^3\Delta(1-\alpha)y_3 + \epsilon^3\Delta(1-\alpha)y_5$$
$$\dot{y}_4 = -y_2, \quad \dot{y}_5 = y_1 + y_6, \quad \dot{y}_6 = -y_3 - y_5$$
$$\kappa = 2H_2^2/(A\omega_2^2), \quad \Delta = 12\,b\omega_0\kappa$$

The roots of the characteristic equation of system (4.10) are

$$\lambda_{1,2} = \pm\sqrt{3(1-\alpha)} + O(\epsilon^2)$$
$$\lambda_3 = O(\epsilon^4), \quad \lambda_4 = \epsilon^3 \frac{4\Delta(1-\alpha)}{3\alpha(3\alpha - 4)} + O(\epsilon^4)$$
$$\lambda_{5,6} = \epsilon^3 \frac{4\Delta(1-\alpha)^2}{\alpha(3\alpha - 4)} \pm i(\sqrt{3\alpha - 4} + O(\epsilon^2))$$

When $0 < \alpha < 1$ one of the roots λ_1 or λ_2 is positive; if $1 < \alpha < 4/3$ we have $\lambda_4 > 0$, and if $4/3 < \alpha < 2$ the real parts of λ_5 and λ_6 are positive. Consequently, by Theorem 4, for any values of α in the interval from zero to two (with the exception of small neighbourhoods of the points $\alpha = 1$ and $\alpha = 4/3$) the relative equilibrium of the ring in the orbital coordinate frame, with its plane orthogonal to the radius-vector of the centre of mass, is unstable.

The research reported here was supported by the Russian Fund for Fundamental Research (93-013-16257).

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Translated by D.L.